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SUBDIVISION OF MOTIONS AND ASYMPTOTIC METHODS IN THE THEORY OF NONLINEAR OSCILLATIONS

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At the present time the most effective method in the asymptotic theory of nonlinear oscillations is the method of averaging [1]. In the present note we discuss the possibility of a different point of view with regard to asymptotic methods, that of subdivision of motions into fast motions and slow (evolutionary) motions. As will be shown, the method of averaging and also the method related to it of investigating systems with rapidly rotating phase [1, 2] are special cases of the method of subdivision of motions.

Turning to the formulation of the concept of subdivision of motions, we consider at the same time equations of the unperturbed motion

$$du/dt = U(u) \tag{1}$$

and of the perturbed motion

$$dv/dt = U(v) + V(v), \tag{2}$$

where the perturbing function V has an asymptotic expansion in powers of a small parameter $V(\epsilon, v) = \epsilon V_1(v) + \epsilon^2 V_2(v) + \cdots$. We denote by

$$u = u(w, t) \tag{3}$$

the solution of the system (1) with initial conditions $u|_{t=0} = u(w, 0) = w$.

It is not difficult to show that the solution of equation (2) can likewise be written in the form (3), but it is then necessary to regard w not as constant but as depending on t, analogous to the method of variation of parameters in linear equations. The equation for w thus obtained will in general contain the time explicitly. In particular, in the linear case there may appear secular terms. It is clear that the investigation is essentially simplified when the equation for w is autonomous, i.e., does not contain the time explicitly. This equation is naturally called evolutionary, since it describes a slow variation of the system which is independent of its basic rapid motion. In this case we shall say of the system (2) that it admits subdivision of motions.

Systems admitting subdivision of motions play a role among general systems of the form (2) analogous to the role of diagonal systems among linear systems. Just as a system of linear equations can be transformed to diagonal form by a linear change of variables, a nonlinear system (2) can be transformed by a nonlinear change of variables into a form which admits subdivision of motions. The degenerate case, which corresponds to the case of Jordan form in linear equations, is excluded.

Our present goal is to determine those cases in which a system admits subdivision of motions. Substituting (3) into (2) we have

$$\frac{\partial u}{\partial t} + \frac{\delta u}{\delta w}\frac{dw}{dt} = U(u) + V(u).$$

Here and later the symbol $\delta u/\delta w$ and analogous symbols denote matrices whose elements are the partial derivatives of the components of the vector u with respect to the components of the vector w. By the definition of the function $u(w, t)$ its partial derivative with respect to t is $U(u)$. Therefore the first terms in the two sides of the equation cancel, and we obtain after multiplication on the left by $(\delta u/\delta w)^{-1}$

$$dw/dt = W(w,\ t), \tag{4}$$

where we have set

$$W = (\delta u/\delta w)^{-1}\, V. \tag{5}$$

By the definition of a system which admits subdivision of motions, the right member of (4) must not contain the time explicitly. A simple calculation shows that

$$\frac{\partial W}{\partial t} = \left(\frac{\delta u}{\delta w}\right)^{-1} \left[\frac{\delta V}{\delta u}\, U - \frac{\delta U}{\delta u}\, V\right]. \tag{6}$$

Therefore a necessary and sufficient condition for subdivision of motions is

$$\frac{\delta V}{\delta u} U - \frac{\delta U}{\delta u} V = 0. \tag{7}$$

If this is satisfied, equation (2) gives the formula

$$v = u(w(t),\ t), \tag{8}$$

where $w(t)$ is the solution of equation (4), which in this case takes the form

$$dw/dt = V(w). \tag{9}$$

This is obtained at once from (5) for $t = 0$, since then $\delta u/\delta w = E$. But since W does not depend on t, the equation $W = V(w)$ holds identically.

Thus to solve equation (2) when condition (7) is satisfied, it is sufficient to solve equations (1) and (9) independently, and then to substitute in the solution of one of these equations (and it does not matter which, because of the symmetry of condition (7)) the solution of the other in place of the initial conditions (cf. (8)). Note that if U and V are linear functions of their arguments, then (7) is simply the condition for permutability of the matrices U and V. Thus the left member of condition (7) is a natural generalization of the idea of commutator to the case of nonlinear operators.

We now proceed to the proof of the fact that by a change of variables the system (2) can be brought to a form which admits subdivision of motions. The smallness of the parameter ϵ has played no role up till now, but it now acquires decisive value. We will prove the existence of an asymptotic series $y = v + \epsilon Q_1(v) + \epsilon^2 Q_2(v) + \cdots$ such that the equation for y admits subdivision of motions. Simple but lengthy calculations show that an equation is obtained for y analogous to the equation for v, with the same principal term, as one would expect:

$$dy/dt = U(y) + \epsilon Y_1(y) + \epsilon^2 Y_2(y) + \cdots. \tag{10}$$

It can be verified that the coefficient Q_n, which we must determine, enters Y_n in the following manner:

$$Y_n = \tilde{Y}_n + \frac{\delta Q_n}{\delta y}\, U - \frac{\delta U}{\delta y}\, Q_n, \tag{11}$$

where $\tilde{Y}_n$, except for the given functions $V_1, \cdots, V_n$ depends only on the preceding $Q_1, \cdots, Q_{n-1}$. In particular $\tilde{Y}_1$ is simply $V_1(y)$. Therefore when we come to determine Q_n in its turn, $\tilde{Y}_n$ may be considered as a known function of y. We are to select the coefficients Q_n so that equation (10) admits subdivision of motions. As we saw above, this is equivalent to the requirement that the functions Y_n commute with U in the sense of condition (7). Introducing the notation

$$\mathfrak{L}_U(Y) = \frac{\delta Y}{\delta u}\, U - \frac{\delta U}{\delta u}\, Y, \tag{12}$$

we see that the problem of determining Q_n is reduced to the problem of expressing the known function $\tilde{Y}_n$ as the sum of two terms, one of which is in the range of the linear operator $\mathfrak{L}_U$ while the other is annulled by this operator.

$$\tilde{Y}_n = -\mathfrak{L}_U(Q_n) + Y_n, \qquad \mathfrak{L}_U(Y_n) = 0. \tag{13}$$

Strictly speaking, the expression (12) is not an operator, since to specify an operator completely, it is necessary to give its domain. The choice of domain is dictated by the set of functions $\tilde{Y}_n$ which must be decomposed into the sum (13). If the choice of domain is determined in this way or otherwise, the question of the possibility of the decomposition (13) reduces to the question of the absence in the operator $\mathfrak{L}_U$ of Jordan cells corresponding to the eigenvalue zero. We shall not discuss the possibility of degeneracy just now, but rather proceed to the consideration of a practically important case, in which we may not only prove the existence of the decomposition, but may actually construct it.

This decomposition is based on a different interpretation of the decomposition (13), which is obtained in the following manner. The formula (5) uniquely associates with each function $V(u)$ a function of w and t, which is the result of a parallel displacement $V(u)$ along trajectories of the unperturbed motion $u(w, t)$. The decomposition (13) generates a decomposition with a very simple interpretation among such functions. This is the decomposition of an arbitrary function of the form (5) into the sum of two: a function integrable along the trajectories (i.e., representable as the partial derivative with respect to t of a function of the same form), and a function which does not change with displacements along a trajectory. Such an interpretation follows at once from formulas (5), (6), and (13). We note that by virtue of these same formulas and of the relation $\delta u/\delta w = E$ for $t = 0$, which follows from (3), the decomposition of functions of the form (5) gives the decomposition (13) on substitution of $t = 0$.

The interpretation just analyzed is general and always useful, but in one case, important for applications, it leads at once to an effective solution of the problem. This is the case where the displacement $\tilde{Y}_n$ along trajectories generates an almost periodic function of t whose frequencies do not accumulate at zero. (Indeed they exist in the form of just such functions.) The problem is then reduced to separating out from the function its mean value, since it is not hard to verify that the vanishing of the mean value is a necessary and sufficient condition for integrability of such functions (in the sense of preserving inclusion in the class). Thus the decomposition of the function into the sum of a constant plus a function with mean value zero gives the required decomposition. Omitting detailed calculations, we mention the final result. (In the formulas, $u = u(w, t)$, and integration with respect to t is along a trajectory, i.e., with w fixed.)

$$Y_n(w) = \lim_{T\to\infty} \frac{1}{T} \int_0^T \left(\frac{\delta u}{\delta w}\right)^{-1} \tilde{Y}_n(u)\, dt, \tag{14}$$

$$Q_n(w) = \lim_{T\to\infty} \frac{1}{T} \int_0^T (T - t) \left(\frac{\delta u}{\delta w}\right)^{-1} [\tilde{Y}_n(u) - Y_n(u)]\, dt. \tag{15}$$

The formulas obtained are somewhat simplified if $\tilde{Y}_n$ turns out to be a periodic function of t. In this case it is clearly sufficient to take the mean over a period T, even if this period depends on w.

Note that nonuniqueness of the choice of Q_n (since we may add to Q_n an arbitrary term Q'_n for which $\mathfrak{L}_U(Q'_n) = 0$) is removed from formula (15) by the requirement that Q_n have mean zero. This arbitrariness, which is not essential for the construction of the asymptotic theory, may be of importance in the investigation of convergence of asymptotic series. This process of constructing Q_n may turn out not to be very successful, although at first glance it helps the convergence of the series for y as much as possible.

As a second remark we mention the circumstance that the derivation of formulas (14) and (15) was based on the almost periodicity in t of the integrands. The factor $(\delta u/\delta w)^{-1}$ may not be almost periodic in the case of rapidly rotating phase. Certain modifications of the derivation in this case lead to formulas analogous to (14) and (15).

A third and final remark relates to the possibility of generalization of the averaging formulas to the nonperiodic case by analytic continuation of $u(w, t)$ for complex values of t, so that integration takes place along certain curves in the complex domain, along which average value makes sense. Thus, for example, if $U(u)$ is a linear operator with real eigenvalues, formulas (14) and (15) give the desired decomposition by integration along the imaginary t axis. It is of interest to clarify whether such extensions are illusory or whether they essentially extend the circle of problems (13) which admit solutions by formulas of the type (14) and (15). The example of the function $\sinh t + \sin t$ is a small encouragement in this direction.

It is therefore of interest to try to find approaches to the solution of the decomposition problem. We shall discuss one possible approach. The equations of the unperturbed motion look simplest if a system of first integrals of equation (1) is taken for new unknowns. But since the number of first integrals is one less than the order of the system, it is still necessary to adjoin one further independent function, which is naturally called the phase variable. In any case, such functions can be chosen in the neighborhood of an arbitrary regular point of equation (1). It is easy to verify that in these variables the decomposition problem leads to equations which can be integrated by quadratures. The solution obtained contains arbitrary functions. Formally speaking, subdivision of motions takes place for an arbitrary choice of these functions. However, the effectiveness of the asymptotic expansion depends in an essential manner on the boundedness of its coefficients. In some cases the requirement of boundedness of coefficients eliminates the arbitrariness of choice of solution. Lack of space prevents a more detailed discussion in this note of this interesting question, which is probably connected with questions of convergence of asymptotic expansions.

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